Complexity and Expressivity of Propositional Logics with Team Semantics

ESSLLI 2024 course

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Complexity and Expressivity of Propositional Logics with Team Semantics
Arne Meier, Jonni Virtema
6th of August

Lecture 2: Expressive power of team-based logics

Literature: [YV17; Hel+14]

How to characterise expressivity – Tarski's semantics

Definition 11

If φ is formula of propositional logic, with variables $p_1 \ldots, p_n$, one can say that φ defines the n-ary Boolean function $f_\varphi : \{0,1\}^n \to \{0,1\}$ defined

$$s \mapsto s(\varphi),$$

where s is an assignment for the variables $p_1 \ldots, p_n$.

One can then ask, which Boolean functions can be expressed in propositional logic.

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One can then ask, which Boolean functions can be expressed in propositional logic. In fact, propositional logic is expressively complete (in the standard Tarskian setting).

Proposition 12

Every Boolean function can be defined in propositional logic.

How to characterise expressivity – Team semantics – definitions

In team semantics setting, a propositional formula defines a set of teams that satisfy it.

Definition 13

We define

$$\mathrm{Teams}(\varphi) := \{T \mid T \models \varphi\}$$

We then want to know, what are the families of teams that can be written as $\mathrm{Teams}(\varphi)$ by some formula φ.

$$T \models \varphi \iff T\upharpoonright_{VAR(\varphi)} \models \varphi$$

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Definitions of downward/union closure and flatness generalise to families of teams.

Definition 14

A family of teams $\mathcal{T}$ is

- downward closed, if ($T \in \mathcal{T}$ and $S \subseteq T$) implies $S \in \mathcal{T}$.
- union closed, if $T, S \in \mathcal{T}$ implies $T \cup S \in \mathcal{T}$.
- flat, if $T \in \mathcal{T}$ if and only if $\{t\} \in \mathcal{T}$, for all $t \in T$.

Properties of families of teams

$T \models \varphi \iff \forall s \in T \; \{s\} \models \varphi$

Proposition 15

A family of teams $\mathcal{T}$ is flat if and only if it is union & downward closed and $\emptyset \in \mathcal{T}$.

Proof.

Left-to-right direction if trivial.

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Proof.

Left-to-right direction if trivial. For the right-to-left direction, assume that $\mathcal{T}$ is union & downward closed and that $\emptyset \in \mathcal{T}$. Now the left-to-right direction of

$$T \in \mathcal{T} \iff \forall t \in T : \{t\} \in \mathcal{T}$$

follows from downward closure, while the converse direction follows from union closure. The empty team property is required to omit the special case of $\mathcal{T} = \emptyset$. □

Proposition 16

Let $\mathcal{T}$ be a flat family of teams. Then $T \in \mathcal{T}$ if and only if $T \subseteq \bigcup \mathcal{T}$.

Proof.

Left-to-right direction is trivial and follows directly from the definition of a union.

Proposition 16

Let $\mathcal{T}$ be a flat family of teams. Then $T \in \mathcal{T}$ if and only if $T \subseteq \bigcup \mathcal{T}$.

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Right-to-left direction: By Proposition 15, $\mathcal{T}$ is union closed and downward closed.

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From union closure of $\mathcal{T}$ it follows that $\bigcup \mathcal{T} \in \mathcal{T}$.

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Left-to-right direction is trivial and follows directly from the definition of a union. Right-to-left direction: By Proposition 15, $\mathcal{T}$ is union closed and downward closed. From union closure of $\mathcal{T}$ it follows that $\bigcup \mathcal{T} \in \mathcal{T}$. Now since $\mathcal{T}$ is downward closed and $T \subseteq \bigcup \mathcal{T}$, if follows that $T \in \mathcal{T}$. □

How to characterise expressivity – Team semantics – results

We have already seen (and partly proved) the following closure results:

Proposition 17

- *A family of teams defined by a PL-formula is flat.*
- $\mathrm{PL}[\mathrm{dep}]$-*definable team families are downward closed and include the empty team.*

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$p, \neg p$

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$$T \models \varphi \vee \psi \iff T_1 \models \varphi \text{ and } T_2 \models \psi \text{ for some } T_1 \cup T_2 = T$$

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By IH, the right-hand side is equivalent to: $\forall t \in T : \{t\} \models \varphi$ or $\{t\} \models \psi$. This is again equivalent to $\forall t \in T : \{t\} \models \varphi \vee \psi$, due to the empty team property. □

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Interestingly the above results can be strengthened to if and only if!

Expressivity of PL with team semantics

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For every flat family $\mathcal{T}$ there exists a PL-formula φ such that $\mathcal{T} = \mathrm{Teams}(\varphi)$.

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$$\Phi := \bigvee_{s \in \bigcup \mathcal{T}} \varphi_s$$

and claim that $\mathcal{T} = \mathrm{Teams}(\Phi)$.

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Theorem 19

A family of teams is definable in PL if and only if the family is flat.

Expressivity: the downward closed case

Let's consider an extension $\mathrm{PL}[\varovee]$ of PL with the so-called Boolean disjunction

$$T \models \varphi \varovee \psi \text{ if and only if } T \models \varphi \text{ or } T \models \psi.$$

$T \models p \varovee \neg p$

Proposition 20

$\mathrm{PL}[\varovee]$ *is downward closed and has the empty team property.*

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$\mathrm{PL}[\varovee]$ *is downward closed and has the empty team property.* PL(DEP)

It is easy to note that dependence atoms can be expressed in $\mathrm{PL}[\varovee]$:

$$T \models \mathrm{dep}(p_1, \ldots, p_n, q) \text{ if and only if } T \models \bigvee_{b \in \{\bot, \top\}^n} \left(p_1^{b_1} \wedge \cdots \wedge p_n^{b_n} \wedge (q \varovee \neg q)\right),$$

where $p^\bot := \neg p$ and $p^\top := p$.

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Let $\mathcal{T}$ be a family of teams with the aforementioned properties. Define

$$\Phi := \bigovee_{T \in \mathcal{T}} \bigvee_{s \in T} \varphi_s, \text{ where } \varphi_s \text{ is a formula whose only satisfying assignment is } s.$$

$s \models \Phi$

We claim that $\mathrm{Teams}(\Phi) = \mathcal{T}$.

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Can you make the formula a bit shorter?

Types and characterising formulae

We define some auxiliary notation and formulae:

- $\mathrm{Type}_{\Psi}(s) := \{\varphi \in \Psi \mid s \models \varphi\}$, for a set of PL-formulae Ψ and an assignment s.
- For $\Gamma \subseteq \Psi$, define $\theta_{\Gamma} := \bigwedge_{\psi \in \Gamma} \psi \wedge \bigwedge_{\psi \in \Psi \setminus \Gamma} \neg\psi$,

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- $\mathrm{Type}_\Psi(T) := \{\mathrm{Type}_\Psi(s) \mid s \in T\}$, for a team T.

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Lemma 22

Assume that T and S be teams and let Ψ be a finite set of PL-formulae.

1. *For each $\psi \in \Psi$, $T \models \psi$ if and only if $\psi \in \bigcap \mathrm{Type}_\Psi(T)$.*
2. *If $T \models \bigovee \Psi$ and $\mathrm{Type}_\Psi(S) \subseteq \mathrm{Type}_\Psi(T)$, then $S \models \bigovee \Psi$.*

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Case 1. follows by flatness of PL, and 2. uses 1. together with the definition of $\ovee$. Intuitively, it follows due to downward closure.

Consider next the formula stating that the truth value w.r.t. a set of propositions $\Psi \subseteq \text{PROP}$ is constant:

$$\gamma := \bigwedge_{p \in \Psi} \text{dep}(p).$$

$S \models \gamma \vee \gamma$

Hence $T \models \gamma$ if and only if $|\text{Type}_{\Psi}(T)| \leq 1$.

Expressive power of $\mathrm{PL}[\mathrm{dep}]$

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Hence $T \models \gamma$ if and only if $|\mathrm{Type}_{\Psi}(T)| \leq 1$. Define now recursively

$$\gamma^0 := p \wedge \neg p, \qquad \gamma^{k+1} := (\gamma^k \vee \gamma).$$

It is easy to show by induction that $T \models \gamma^k$ if and only if $|\mathrm{Type}_{\Psi}(T)| \leq k$.

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If $\Psi \subseteq \mathrm{PROP}$ *is a finite set of propositions and* $T \neq \emptyset$ *a team, there is a* $\xi_T \in \mathrm{PL}[\mathrm{dep}]$ *s.t. for every* S

$$S \models \xi_T \iff \mathrm{Type}_{\Psi}(T) \not\subseteq \mathrm{Type}_{\Psi}(S).$$

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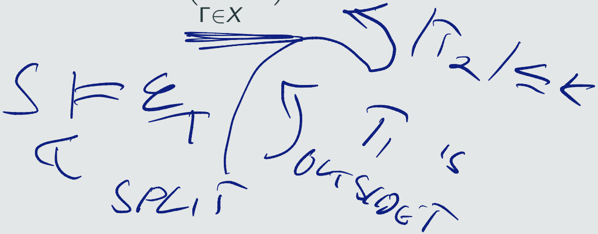
Let $|\mathrm{Type}_\Psi(T)| = k + 1$. Recall θ_Γ is a characterisic formula of Γ. We define

$$\xi_T := \left(\bigvee_{\Gamma \in X} \theta_\Gamma \right) \vee \gamma^k, \text{ where } X = \mathcal{P}(\Psi) \setminus \mathrm{Type}_\Psi(T).$$



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Now given a team S we have

$$\begin{aligned} S \models \xi_T &\iff \text{there are } T_1, T_2 \text{ s.t. } T_1 \cup T_2 = S, \mathrm{Type}_\Psi(T_1) \subseteq X, |\mathrm{Type}_\Psi(T_2)| \leq k \\ &\iff |\mathrm{Type}_\Psi(T) \cap \mathrm{Type}_\Psi(S)| \leq k \\ &\iff \mathrm{Type}_\Psi(T) \not\subseteq \mathrm{Type}_\Psi(S). \quad \square \end{aligned}$$

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$\mathrm{PL}[\varovee] \leq \mathrm{PL}[\mathrm{dep}]$ direction: Let $\varphi = \bigcirc\!\!\!\!\vee\, \Psi$ be a $\mathrm{PL}[\varovee]$-formula in a normal form, where $\Psi \subseteq \mathrm{PL}$. Define

$$\eta := \bigwedge_{T \notin \mathrm{Teams}(\varphi)} \xi_T, \text{ where } \xi_T \text{ is as in Lemma 23.}$$

Intuitively $S \models \eta$ iff no falsifying team of φ is completely subsumed by S.

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By definition η is a $\mathrm{PL}[\mathrm{dep}]$-formula. To prove that $\mathrm{Teams}(\eta) = \mathrm{Teams}(\varphi)$, assume first that $S \in \mathrm{Teams}(\varphi)$, and consider any $T \notin \mathrm{Teams}(\varphi)$.

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Assume then that $S \notin \mathrm{Teams}(\varphi)$. Since $\mathrm{Type}_\Psi(S) \subseteq \mathrm{Type}_\Psi(S)$, it follows from Lemma 23 that $S \not\models \xi_S$. Thus $S \notin \mathrm{Teams}(\eta)$. □

Dimensions of team families

Definition 25

The lower dimension $\dim(\varphi)$ of a formula φ to is the least n such that

$$T \models \varphi \iff S \models \varphi \text{ for all } S \subseteq T \text{ s.t. } |S| \leq n.$$

The lower dimension of a flat formula is 1, and for a dependence atom it is 2. The lower dimension is not easy to approximate compositionally,

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The lower dimension $\dim(\varphi)$ of a formula φ to is the least n such that

$$T \models \varphi \iff S \models \varphi \text{ for all } S \subseteq T \text{ s.t. } |S| \leq n.$$

The lower dimension of a flat formula is 1, and for a dependence atom it is 2. The lower dimension is not easy to approximate compositionally, for that we define the notion of upper dimension. Define $M(\varphi)$ as the set of subset maximal teams satisfying φ.

Definition 26

The upper dimension $\mathrm{Dim}(\varphi)$ of a formula φ is the cardinality of $M(\varphi)$.

Interestingly, $\mathrm{Dim}(\varphi)$ can be given sharp compositional estimates, and it can be shown that $\dim(\varphi) \leq \mathrm{Dim}(\varphi)$.

Estimates for the upper dimension

Lemma 27

We have the following upper dimension estimates for $\varphi, \psi \in \mathrm{PL}[\mathbin{\varovee}]$:

1. $\mathsf{Dim}(p) = \mathsf{Dim}(\neg p) = 1$.
2. $\mathsf{Dim}(\varphi \wedge \psi) \leq \mathsf{Dim}(\varphi)\,\mathsf{Dim}(\psi)$.
3. $\mathsf{Dim}(\varphi \vee \psi) \leq \mathsf{Dim}(\varphi)\,\mathsf{Dim}(\psi)$.
4. $\mathsf{Dim}(\varphi \mathbin{\varovee} \psi) \leq \mathsf{Dim}(\varphi) + \mathsf{Dim}(\psi)$.

Proof.

We omit the cases for (1) and (3), since (1) is trivial, and (3) is analogous to (2).

Estimates for the upper dimension

Lemma 27

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4. $\mathsf{Dim}(\varphi \varovee \psi) \leq \mathsf{Dim}(\varphi) + \mathsf{Dim}(\psi)$.

Proof.

We omit the cases for (1) and (3), since (1) is trivial, and (3) is analogous to (2). We defer the proof of (2) to the lecture notes.
Case (4): For the Boolean disjunction, it holds that

$$M(\varphi \varovee \psi) \subseteq M(\varphi) \cup M(\psi)$$

and the right-hand side of the inclusion generates the family $\mathrm{Teams}(\varphi \varovee \psi)$. The dimension estimate follows immediately. □

What are dimensions good for?

Proposition 28

$\mathrm{Dim}(\mathrm{dep}(p_1, \ldots, p_n, q)) = 2^{2^n}$.

Proposition 29

For $\varphi \in \mathrm{PL}[\oslash]$, $\mathrm{Dim}(\varphi) \leq 2^k$, where k is the number of occurrences of $\oslash$ in φ.

What are dimensions good for?

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Proposition 29

For $\varphi \in \mathrm{PL}[\varovee]$, $\text{Dim}(\varphi) \leq 2^k$, *where* k *is the number of occurrences of* $\varovee$ *in* φ.

Theorem 30

Let $\varphi \in \mathrm{PL}[\varovee]$ *such that* $\mathrm{Teams}(\varphi) = \mathrm{Teams}(\mathrm{dep}(p_1, \ldots, p_n, q))$. *Then* φ *contains more than* 2^n *symbols.*

Proof.

By Prop 28, $\text{Dim}(\varphi) = \text{Dim}(\mathrm{dep}(p_1, \ldots, p_n, q)) = 2^{2^n}$. Thus $2^{2^n} \leq 2^{\mathrm{occ}_{\varovee}(\varphi)}$ by Prop. 29, implying $2^n \leq \mathrm{occ}_{\varovee}(\varphi)$. Hence φ has at least 2^n Boolean disjunctions. □

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Thus, any translation from $\mathrm{PL}[\mathrm{dep}]$ to $\mathrm{PL}[\varovee]$ leads to an exponential blow-up.

Expressivity of propositional inclusion logic

Theorem 31

A family of teams is definable in $\mathrm{PL}[\subseteq]$ *if and only if it is union closed and includes the empty team.*

Proof.

We will omit the proof, which combines ideas from the characterisation of $\mathrm{PL}[\varovee]$ and its equivalence with $\mathrm{PL}[\mathrm{dep}]$. The result was first shown in [HS15]. □

Conclusion of Lecture 2

- Properties of families of teams.
- Expressivity characterisation of $\mathrm{PL}[\varovee]$.
- Equivalence of $\mathrm{PL}[\varovee]$ and $\mathrm{PL}[\mathrm{dep}]$.
- Expressivity characterisation of $\mathrm{PL}[\subseteq]$.

Complexity and Expressivity of Propositional Logics with Team Semantics
Arne Meier, Jonni Virtema
7th of August

Lecture 3: Inclusion Logic

Literature: [Hel+19; Hel+20]

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